1 General Questions

1. WHY DO I HAVE TO LEARN MATHEMATICS?

The "why" question is perhaps the one encountered most frequently. It is not a matter of if but when this question comes up. And after high school, it will come up again in different formulations. (Why should I become a mathematics major? Why should the public fund research in mathematics? Why did I ever need to study mathematics?) Thus, it is important to be prepared for this question. Giving a good answer is certainly difficult, as much depends on individual circumstances.

First, you should try to find an answer for yourself. What was it that convinced *you* to study and teach mathematics? Why do *you* think that mathematics is useful and important? Have *you* been fascinated by the elegance of mathematical reasoning and the beauty of mathematical results? Tell your students. A heartfelt answer would be most credible. Try to avoid easy answers like "Because there is a test next week" or "Because I say so," even if you think that the question arises from a general unwillingness to learn.

It is certainly true that everybody needs to know a certain amount of elementary mathematics to master his or her life. You could point out that there are many everyday situations where mathematics plays a role. Certainly one needs mathematics whenever one deals with money—for example, when one goes shopping, manages a savings account, or makes a monthly budget.

Nevertheless, we encounter the "why" question more frequently in situations where the everyday context is less apparent. When students ask this question, it should be interpreted as a symptom indicating that they do not appreciate mathematics. This could have many reasons, but the common style of instruction certainly has an influence. Are we really doing our best to make sure that education in the classroom becomes an intellectually stimulating and pleasurable experience? If lessons can be attended without fear of failure and humiliation, it is more likely that mathematics gets a positive image. Try to create an atmosphere where curiosity will be rewarded and where errors and mistakes are not punished but rather are welcomed as a necessary part of acquiring competence. It is most important that you always try to enrich your lessons with interesting facts, examples, and problems showing that mathematics is fascinating and intriguing, and well worth the effort.

It is wrong to say that studying mathematics should always be fun and entertaining. Mathematics, like most worthwhile things, requires a great deal of effort to master. Mathematics is the science of structured thinking, logical reasoning, and problem solving. It requires commitment and time to acquire these skills. To solve a given problem, students have to concentrate on a task, devote attention to details, keep up the effort for some time, and achieve understanding. But the students will be rewarded: Practicing precise logical thinking as well as learning many problem-solving techniques will be useful for many different situations in many different aspects of one's life.

Often, the "why" question originates in a basic misunderstanding of what mathematics is about. The basic fact is that mathematics is useful because it solves problems. In fact, it has been developed for at least 4,000 years to solve problems of everyday life. In early times, mathematics was needed for trading, managing supplies, distributing properties, and even describing the motion of the stars and planets to create calendars and predict seasons for agricultural and religious activities.

Over time, the scope of the problems that can be solved by mathematics has widened considerably, and presently it encompasses all fields of human knowledge. Mathematics is not only useful for measurements and statistics; it also, in particular, is needed to formulate and investigate the laws of nature. With the help of computer technology, mathematics can deal even with very complicated real-world problems. Therefore, mathematical models provide us with useful and vital information about climate change, economic trends and predictions, financial crises, movements of the planets, and the workings of the human body, to name a few. Mathematics has played a major role in many technical developments; a few more recent examples are space exploration, CD players, mobile phones, Internet technologies (e.g., the compression algorithms used for storing music, pictures, and movies), and global positioning system technology for navigation.

So, one of the main reasons to learn mathematics is that it is useful. Today it is more useful than ever before, and it is of importance to more fields of knowledge than ever before. Correspondingly, mathematics is used in many different jobs by scientists, engineers, computer programmers, investment bankers, tax accountants, and traffic planners, to name just a few. Refusing to learn mathematics would mean closing off many career opportunities. Students will perhaps understand that it is important to keep their options open.

However, the comparatively simple mathematics problems prevailing in school are bound to create a wrong impression of the importance of mathematics in the modern world. For students, it may be impossible to understand what calculus has to do with meteorology or risk analysis or automotive engineering, unless you make some attempts to explain these connections. You should point out, for example, that the derivative of a function could be used to describe a rate of change. Differential calculus would be needed where we need precise information about the rate of change of some observable quantity. It should therefore become clear that even the basics of science and technology cannot be understood without a solid background in math.

There is a final, and perhaps the most important, reason why one should learn math, although it is difficult to communicate: Mathematics is a huge, logically and deductively organized system of thought, created by countless individuals in a continuous collective effort that has lasted for several thousand years and still continues at breathtaking pace. As such, mathematics is the most significant cultural achievement of humankind. It should be a natural and essential part of everyone's general education.

2. IS THERE A LANGUAGE CONNECTION BETWEEN MATHEMATICAL TERMS AND COMMON ENGLISH WORDS?

Many mathematical terms are seen by students as words whose definitions must be memorized. Students rarely see applications of these words outside their mathematical context. This is akin to having someone learn words from another language simply for use in that language and then avoiding tying the words back to their mother tongue, even when possible. To learn the meanings of the frequently used mathematical terms without connecting them back to common English usage deprives students of a genuine understanding of the terms involved and keeps them from appreciating the richness and logical use of the English language.

The term *perpendicular*, which everyone immediately associates with geometry, is also used in common English, meaning "moral virtue, uprightness, rectitude." The rays that emanate from the center of a circle to its circumference take on a name closely related, *radius*.

The rectangle is a parallelogram that stands erect. And the right angle, whose German translation is *rechter Winkel,* shows us the connection to the word rectangle. By itself, the word *right* has a common usage in the word *righteous,* meaning "upright, or virtuous."

The word *isosceles* typically is used in connection with a triangle or a trapezoid, as in *isosceles triangle* and *isosceles trapezoid*. The word *isosceles* evolved from the Greek language, with *iso* meaning "equal" and "skelos" meaning "leg." The prefix *iso* is also used in many other applications, as in isomorphism (meaning equal form or appearance, and in mathematics, two sets in a one-to-one onto relation that preserves the relation between elements in the domain), *isometric* (meaning equal measure), or *isotonic* (in mathematics sometimes used as isotonic mapping, referring to a monotonic mapping, and in music meaning that which is characterized by equal tones), just to name a few.

The word *rational* used in the term *rational number* (one that can be expressed as a ratio of two integers) means "reasonable" in our common usage. In a historical sense, a rational number was a "reasonable number." Numbers such as $\sqrt{2}$ were not considered as "reasonable" in the very early days of our civilization, hence the name irrational. The term *ratio* comes from the Latin *ratio,* meaning "a reckoning, an account, a calculation" from where the word rate seems to emanate.

It is clear, and ought to be emphasized, that natural numbers were so named because they were the basis for our counting system and were "nature's way" to begin in a study or buildup of mathematics. Real numbers and imaginary numbers also take their mathematical meanings from their regular English usage.

Even the properties called *associative, commutative*, and *distributive* have the meaning that describes them. The associative property "associates" the first pair of elements together (out of three elements) and then associates the second pair of these three elements. The commutative property changes position of the first element of two to the second position, much as one "commutes" from the first position (at home) to the second position (at work), and at the end of the workday, reverses the "commute" from work back to home. The distributive property "distributes" the first element to the two other elements.

It is clear that when we speak of *ordinal numbers,* we speak of the position or order of a number, as in first, second, third, and so on. The *cardinal numbers* refer to the size or magnitude of a number—that is, its importance. In this sense, one might say that the larger the number is, the more "important" it is. In English, we refer to something of "cardinal importance."

From the word *complete,* we get the term *complementary,* as a complementary item completes something. In its most common usage, an angle is the complement of another if it "completes" a right angle with it.

Once the prefix *con* meaning "with" is understood, terms such as *concentric* (circles having the same center), *coplanar* (in the same plane), or *concyclic* (points lying on the same circle) are easily defined. So just pointing out to students these small hints gives the mathematics terms more meaning.

The word *cave,* a hollow in a mountain, is related to the term *concave*, and one can draw a similar mental picture. The term *convex*, stemming from the Latin *convexus,* meaning "vaulted, arched, . . . drawn together to a point," refers to the opposite of concave.

Some words speak for themselves, such as a *bisector*, which sections something into two (equal) parts. The use of the prefix *bi* in mathematics is usually clear. For example, we have *biconditional* (conditional in two directions), *binomial* (having two names or terms), or *bilateral* (two sided). The word *triangle* also is self-descriptive: "three-angled polygon."

When one considers the definition of *factor* in its common English usage (i.e., "one of the elements contributing to a particular result or situation"), the mathematical definition of factor becomes clear (i.e., one of the numbers multiplied to get a product).

If we stretch the imagination just a bit, the term *fraction* also makes sense. It comes from the Latin *fractio,* meaning "a breaking in pieces," which is what a fraction represents: a piece. In some languages, such as German, a language with roots similar to those of English, the word for *fraction* is "Bruch," which also means "a break or piece," just as the Latin derivation of fraction does. When we fracture a bone, we break the bone.

In making students aware of the words used in mathematics, you also should make them aware of the prefixes that indicate magnitude, such as poly, bi, semi, tri, quad, pent, hex, sept, oct, non, dec, dodec, and so on. These prefixes, when combined with suffixes such as gon, hedron, and so on, allow students to determine a word's meaning.

Not to be overlooked is the term *prime number*, since it stems from the true definition of the word prime. As in the word *primitive,* referring to the basic elements, in a mathematical sense a prime number is one of the basic numbers from which, through multiplication, we build the other numbers.

Whenever a new word or term is introduced in mathematics, it should be related back to common English usage. This may require having a good dictionary at ready reference. The time it takes to tie mathematical terms back to ordinary English will help strengthen the mathematical understanding as well as enlarge a student's regular vocabulary. It is time well spent!

3. HOW MANY LEAVES ARE ON A TREE?

At first sight, this question does not seem to have much to do with mathematics. But it is about counting, and this is where mathematics starts. There is no need for special knowledge in biology, but it could be helpful to join forces with the biology teacher when attacking a problem like this in class.

First, we need to know what type of answer is expected. For a typical tree during the summer, the result will certainly be a very large number. It is fairly hopeless to determine this number exactly by counting the leaves one by one. And indeed, an exact answer (like 51,641) does not appear to be very useful. We would certainly be happy with a rough estimate like "about 50,000." Special techniques of estimating and guessing are needed to obtain a plausible answer, and it is well worth training these abilities, as they are important for applied science and technology.

There are several possible approaches to this problem, and for all of these approaches it is best to consider a real tree and eventually perform measurements on that tree, as it is difficult to obtain good estimates from pictures or memory alone.

One might start to estimate or count the number L of leaves on a typical twig. Next, we need to know the approximate number T of twigs on a branch, and a typical number B of branches on a main branch. Finally, we would count the number M of main branches of the tree. Then a reasonable guess for the number of leaves would be $L \cdot T \cdot B \cdot M$. Of course, the numbers will differ from branch to branch and from tree to tree. And it will be worth discussing what we mean by "typical values" for L, T, B, and M. Indeed, what is a typical branch? One could try to obtain several samples from a tree, count the number of leaves on each of these branches, and then compute the average (arithmetic mean). Depending on the situation and depending on whether you choose lower or higher numbers for these quantities, you will get probably something between 10,000 and 1,000,000 leaves. This should give you an idea of the order of magnitude (i.e., the power of 10) to be expected. It is a typical feature of this type of question that, no matter how we proceed, the final result will be far from unique. After all these estimates, we cannot say whether the result is right or wrong; we can at best judge the result as more or less plausible.

A rather ingenious method to estimate the number of leaves on a tree arises from some insight into the function of the leaves. Their main purpose is to collect sunlight for the process of photosynthesis. Therefore, the leaves are positioned to collect light from all directions, or at least from above. Assuming, for the sake of argument, that the crown of the tree is roughly a sphere, it would make sense to position the leaves in such a way that every point of the surface of this sphere (or at least the upper half of the sphere) is covered by one leaf (or perhaps by two leaves, but not many more, because the leaves outside would cast shadow on the leaves inside, thus making them rather useless for photosynthesis). So we could ask how many leaves are needed to cover the surface of a sphere that has the same diameter as the crown of our tree. To estimate the total number of leaves, we would (a) compute the size of the spherical surface exposed to the sun in square meters and

(b) determine the number of leaves needed to cover 1 square meter with not too much overlap (this can be determined experimentally or from an estimate of the surface area of one leaf, which is an interesting problem in itself).

As an example, consider a tree whose crown has a diameter of about 10 meters. For a sphere with a radius of $r \approx 5$ meters, we get a surface of $4\pi r^2 \approx 314$ m^2. Assuming, that about 1,000 leaves are needed to cover a square meter, we would need about 314,000 leaves to cover the surface of the crown.

There is another, related method. When the tree sheds its leaves in fall, we expect that the ground beneath the crown would be covered by leaves, probably by more than one layer. Assuming that the roughly spherical area under the crown (area $\pi r^2 \approx 78.5$ m^2) is covered by four layers of leaves, we would get the same answer as above. Forestry scientists use a measure called Leaf Area Index (LAI). This is defined as the leaf area (one-sided) in the part of the crown above a unit of ground surface area. So LAI = 1 means that the leaves of the tree can be used to cover the ground just once, while LAI < 1 means that you can still see the ground between the leaves. LAI = 4 means that you can cover the area beneath the crown with four layers of leaves from that tree. In an oak wood, the LAI is typically 5 to 7; for a beach wood, the LAI is 6 to 8. Assuming a LAI of 6, a big old oak tree whose crown has a radius of 15 m could well have a total leaf surface of about $6 \times 15^2 \times \pi \approx 4{,}250$ m^2. If the area covered by one big oak leaf is about 50 cm^2, we would need about 200 leaves to cover 1 m^2 of ground, and therefore we get about $200 \times 4{,}250 = 850{,}000$ leaves for that oak tree.

Questions like this have come to be known as Fermi questions, named after the famous physicist and Nobel laureate Enrico Fermi, who liked to pose questions like “How many piano tuners are in Chicago?” to train his students’ abilities to seek fast, rough estimates in situations where the available facts are incomplete or where a direct measurement seems to be difficult or impossible. Other examples of Fermi questions include the following:

“How many hairs are on your head?”

“How many drops of water are in the Atlantic ocean?”

“How many words have you said so far in your life?”

“How many golf balls will fit into your classroom?”

“How long is the queue if all the people living in New York would line up?”

“What is the weight of the U.S. public debt in 100 dollar bills?” and so on.

When approaching these questions, it is important to avoid giving up early with a shrug and the attitude “How could I know?” The main effect is that students will have to create a strategy, think in big numbers, learn to change units as needed, and learn to make estimates and reasonable guesses. In short, they are forced to employ everyday reasoning when doing mathematics.

4. WHY DO WE HAVE TO LEARN ABOUT THE HISTORY OF MATHEMATICS?

Students study a variety of mathematical topics throughout their secondary school experience. They learn about the Pythagorean theorem, Pascal's triangle, Euclidean geometry, the Cartesian plane, Platonic solids, Boolean algebra, Gaussian arithmetic, Diophantine equations, Euler's equation, Fibonacci numbers, and a multitude of other mathematical concepts and principles. You may notice that the 10 aforementioned topics are named after great mathematicians, whose work and discoveries contributed to the growth and development of mathematical ideas.

It is essential for students to go beyond the rote and procedures of learning mathematical skills in the classroom. Discovering where all of these mathematical rules, axioms, conjectures, and theorems came from will help reveal to students how the developments in mathematics have helped to shape the cultures of the world, and how the cultures of the world influenced mathematics, for example, how the growth of numbers and numeration is revealed in language, how the concept of quantification led to different theories of arithmetic, how the concept of informal measurement led to geometry, how early theories of astronomy led to trigonometry and logarithms, and how analytic geometry and calculus were developed to gain a better understanding of motion.

One reason why students should learn mathematics from a historical perspective is that mathematics has a rich cultural heritage. Studying the history of mathematics will help students appreciate this culture. Moreover, learning mathematics from a historical perspective will help students realize the connections of mathematics to other disciplines such as art, music, architecture, crafts, religion, and philosophy.

Learning about the history of mathematics can give students a deeper appreciation of mathematics, and therefore help them personalize mathematics. In addition, by learning about the great mathematicians, students can appreciate the contributions of different cultures throughout the world. Students can also learn to value the persistent efforts and collected genius of these influential mathematicians, and subsequently discover how mathematics evolved throughout the ages and how new branches of mathematics were developed. Moreover, students will be able to recognize the value of how the combined advances in mathematics transformed not only mathematics but also other sciences and civilization.

Learning about the history of mathematics motivates students and the mathematics becomes more functionally relevant to their lives, and in turn, they will become more interested, which helps to spur student achievement. Oftentimes, students take a small tidbit of historical facts revealed in the mathematics classroom to their history class, thereby forming a genuine link between the subjects. Each subject benefits from this connection.

5. WHO INTRODUCED THE HINDU-ARABIC NUMBERS TO THE WESTERN WORLD, AND WHEN?

In the early Middle Ages, Europeans still used the Roman numeral system for commerce and trade. The Roman numeral system was quite limited in that it lacked a real system of place value as well as a value of zero. There were only seven symbols, and each represented a different number:

I = 1

V = 5

X = 10

L = 50

C = 100

D = 500

M = 1,000

Numbers were read from the left to the right, with the largest valued symbol appearing in the leftmost position. In addition, there was a minimal placement agreement that if a smaller valued symbol preceded a larger valued symbol, one would subtract the smaller value from the adjacent larger value. For example, the year 2013 would be represented as MMXIII and the year 1942 would be MCMXLII. The only additional representation this system offered was that if a horizontal line was placed over a symbol, it was meant to multiply its face value by 1,000, so $\overline{\text{V}}$ would represent 5,000.

We retain Roman numerals for special uses today such as the numbering of preface pages in texts, memorializing the construction date of significant buildings, naming subsequent heirs to a throne or members of a family such as John Smith III, and recording the release dates for movies.

When working in Roman numerals, one would often need to do calculations on an abacus. You can imagine how long a process would be needed for very large numbers that are common today, such as the U.S. national debt, which as of the beginning of the year 2013 stood at $16,436,821,542,578 and is still growing.

One of the most famous mathematicians of the Middle Ages was Leonardo of Pisa (also known as Fibonacci; c. 1170–1250). He was born in Italy but grew up in North Africa, near Algiers, where his father, Guglielmo, was stationed as a trade representative for the city-state of Pisa. He travelled widely with his father in the Middle East and was exposed at an early age to the Arab system of numeration and the use of zero. He studied under the leading Arab mathematicians and quickly realized the advantage of the Hindu-Arabic place-valued decimal system over the Roman numeral system. In 1202, at the age of 32, Fibonacci wrote *Liber Abaci (The Book of Calculation),* which was well received by educated Europeans. It introduced them not only to the Hindu-Arabic system and the number zero but also to our current notations for fractions and square roots.

The Hindu-Arabic numeration system had 10 symbols, which have been modified and morphed from their appearances in the Middle Ages into our current mathematical number symbols. See the comparative chart of the 10 symbols from medieval times to modern times below.

Hindu-Arabic	٠	١	٢	٣	٤	٥	٦	٧	٨	٩
European	0	1	2	3	4	5	6	7	8	9

By using the 10 symbols called *digits* (0, 1, 2, 3, 4, 5, 6, 7, 8, 9) and a place-value system based on the powers of 10, numbers of any size could easily be represented and used in calculations.

The place-value system permitted the digits to carry the “weight of their place” times their face value. For example, in the number 123,056, we note the place value below the digit face value, beginning with the rightmost digit.

1	2	3	0	5	6
Hundred-thousands	Ten-thousands	Thousands	Hundreds	Tens	Ones
10^5	10^4	10^3	10^2	10^1	10^0

To evaluate, start from the right end and the first digit, 6:

Digit 6 is in the units or ones column and represents	$6 \times 1 = 6$
Digit 5 is in the tens place and represents	$5 \times 10 = 50$
Digit 0 is in the hundreds place and represents	$0 \times 100 = 000$
Digit 3 is in the thousands place and represents	$3 \times 1{,}000 = 3{,}000$
Digit 2 is in the ten-thousands place and represents	$2 \times 10{,}000 = 20{,}000$
Digit 1 is in the hundred-thousands place and represents	$1 \times 100{,}000 = 100{,}000$
Adding all these values gives us the original number	123,056

There is no limit to the size of the number you can represent using only the 10 digits and employing the zero as a placeholder.

The number above represents a whole number. If we expand to the right of the decimal point, the system works perfectly as digits that appear there have a place value continuing in order from left to right as 10^{-1}, 10^{-2}, 10^{-3} as tenths, hundredths, thousandths, and so on.

Acceptance and use of the Hindu-Arabic number system quickly spread and became practical and functional in everyday life as well as in spheres of trade and finance such as importing and exporting, banking, calculating interest, and money exchanges. These changes had a profound impact on all mercantile and merchandising activities throughout the European world.

6. WHAT ARE THE THREE FAMOUS PROBLEMS OF ANTIQUITY?

Unsolved problems have always fascinated people. The three geometric problems of antiquity are perhaps the most famous of all. They were posed by the ancient Greeks and remained unsolved for more than 2,000 years. The solution obtained in modern times, with the help of abstract algebra, was not quite satisfactory either.

The three problems are as follows:

1. *The trisection of an arbitrary angle*

   Given an angle, construct an angle one-third its measure.

2. *The doubling of a cube*

   Given a cube, construct a cube with double the volume.

3. *The squaring of a circle*
   Given a circle, construct a square that has the same area as the circle.

But here is a hitch. The only tools allowed to accomplish these tasks are a compass and an unmarked straightedge (and a pencil, of course). Moreover, according to Euclidean geometry, the only operations that are allowed to be done with these tools are the following:

1. Drawing points
2. Connecting two points with a line segment
3. Drawing a circle centered at a given point with a given line segment as radius
4. Marking intersection points (of two lines, of a line and a circle, of two circles)

When we say that something can be done with compass and unmarked straightedge, we mean that the whole construction can be reduced to a finite sequence of the four steps listed above.

It turned out that the three problems of antiquity are impossible to accomplish with a compass and straightedge, and using only allowed operations. For a very similar reason, it is impossible to construct a regular heptagon (a seven-sided polygon).

The impossibility of these constructions follows from modern algebra and is rather difficult to prove at secondary-school level. The general result is the following:

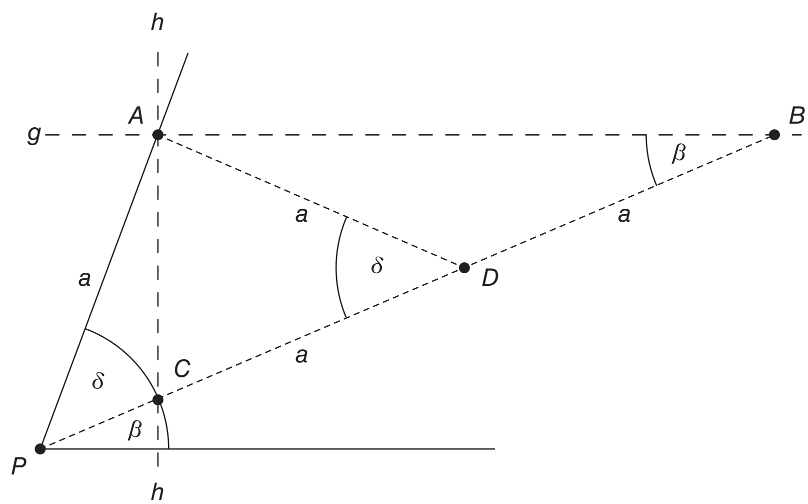
Starting with a line segment of unit length, a line segment of length L can only be constructed (using compass and unmarked straightedge) if L can be obtained from the rational numbers by a finite number of steps involving the operations addition/subtraction, multiplication/division, and taking square roots. Using the theory of fields, one can show that whenever a number L is constructible in this sense, then L is an algebraic number (i.e., a root of a polynomial with integer coefficients), and that the degree of its minimal polynomial is a power of 2 (the minimal polynomial is the polynomial with the smallest degree that has L as a root). It can be seen that problems 1 and 2 lead to minimal polynomials of degree 3 and problem 3 would need the construction of the square root of π, which is a transcendental number (i.e., it is not the root of a nontrivial polynomial with integer coefficients).

Thus, all of these problems have been proven to be unsolvable. Of course, this result depends on the required usage of compass and unmarked straightedge only for its construction. It is possible to obtain solutions, if other tools are allowed or if the tools are used in nonstandard ways.

Here we will provide an example for a trisection of an angle that was described by Pappus around AD 320 but is certainly much older (probably the oldest known construction of this type).

We are given an angle with vertex at *P*, as indicated in Figure 1.1. Draw a point *A* on one of the legs of the angle, thus creating a line segment *PA* of length *a*. Through point *A*, draw one line *g* parallel to the other leg of the angle, and one line *h* perpendicular to *g*.

Figure 1.1



Next, draw a line through the vertex *P* intersecting *h* at *C* and *g* at *B* in such a way that the length of the segment *CB* is precisely 2*a*. (This step can be achieved if we mark the length 2*a* on a ruler and move the ruler in the plane until it has the required position—unfortunately, this is not an allowed Euclidean operation.)

The line segment *PB* cuts the given angle into two parts, which we call β and δ. Obviously, the angle at B also equals β.

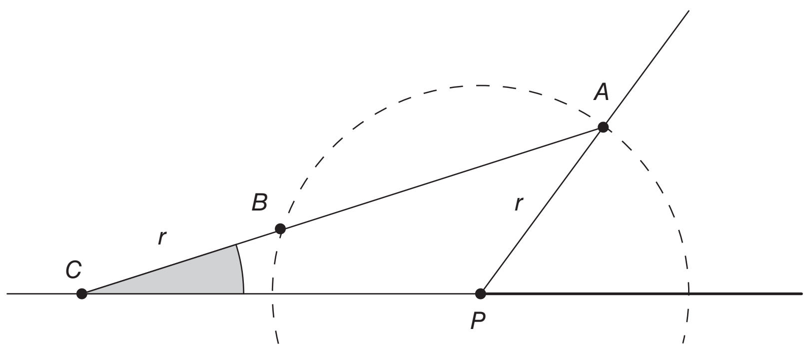
The midpoint *D* of segment *BC* defines an isosceles triangle *DAP* so that the angle δ also appears at *D*, as indicated. But the triangle *ADB* is also isosceles so that the angle δ is easily shown to be 2β. Hence the original angle at *P* is $\beta + \delta = 3\beta$. This shows that the newly constructed angle β trisects the given angle at *P*.

Another method was developed by Archimedes: Given an angle with vertex *P*, draw a circle with radius *r* around *P*. Let *A* be the intersection of that circle with one of the legs of the given angle. Through *A*, draw a straight line as in Figure 1.2, such that the segment *CB* has precisely the length *r*.

Then the indicated angle at *C* is precisely one-third of the given angle at *P*. This is rather easy to see and probably a good exercise for your students.

Fitting a line segment of a given length between two given lines in a certain way is called a "Neusis construction." It requires that two points on a straightedge are marked, and the straightedge is moved until the two points have the desired position. Unfortunately, this is not an allowed Euclidean operation.

Figure 1.2



These famous problems have attracted amateurs who tried to find solutions, ignoring the proven fact that they cannot be solved. Even today, mathematicians are frequently approached by people who believe that they have a solution to one of these famous problems. What they really have is most probably either an approximate or a Neusis construction.

7. WHAT ARE THE FIBONACCI NUMBERS?

Leonardo of Pisa, called Fibonacci (1170–1250), was one of the best-known mathematicians of the Middle Ages. He lived in Bugia, a trading post in Algiers, where his father was probably the resident trade authority for the city-state of Pisa. Fibonacci travelled widely with his father and was exposed to Eastern mathematical thoughts and ideas, and he carried the accomplishments of Persian (Iranian), Indian, and Arabic mathematicians back to Western civilization. One of the most well-known examples of Eastern mathematics he worked with and introduced was the number sequence that bears his name.

The Fibonacci sequence[1] is defined by giving its initial value(s) and a rule that determines the next term in the sequence. The first two terms of the Fibonacci sequence are

$$F_1 = 1$$

$$F_2 = 1$$

Fibonacci Rule: Each proceeding [next] term is the sum of the previous two terms so that

$$F_3 = F_2 + F_1 = 1 + 1 = 2$$

$$F_4 = F_3 + F_2 = 2 + 1 = 3$$

$$F_5 = F_4 + F_3 = 3 + 2 = 5$$

$$F_6 = F_5 + F_4 = 5 + 3 = 8$$

And, in general: $F_n = F_{n-1} + F_{n-2}$ with $n \geq 3$

Obtaining the next dozen or so terms is usually easy for students, but it may be interesting to discuss whether there will be even and odd numbers, squares, cubes, primes, and composites coming up in the sequence. They quickly get 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, and they will ask, "So, what is so special about this sequence that it has enjoyed such long-lasting popularity?" The answer is to demonstrate the amazing beauty and wealth of applications that developed from this simple sequence known to Indian mathematicians in the sixth century but popularized in European culture by Fibonacci in his well-received book *Liber Abaci* (1202).

The Fibonacci numbers emanate from the discussion of the "Growth of a Pair of Rabbits," which appears in Chapter 12 of Fibonacci's book *Liber Abaci*: Here is a translation of the problem as it appeared in Fibonacci's book.

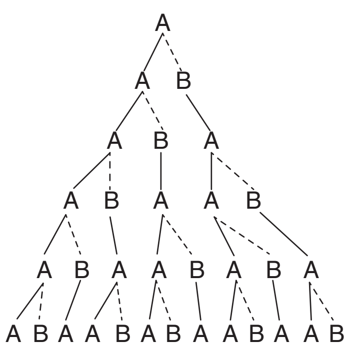
Beginning 1 First 2 Second 3 Third 5 Fourth 8 Fifth 13 Sixth 21 Seventh 34 Eighth 55	**"A certain man had one pair of rabbits together in a certain enclosed place, and one wishes to know how many are created from the pair in one year when it is the nature of them in a single month to bear another pair, and in the second month those born to bear also. Because the above written pair in the first month bore, you will double it; there will be two pairs in one month. One of these, namely the first, bears in the second month, and thus there are in the second month 3 pairs; of these in one month two are pregnant and in the third month 2 pairs of rabbits are born and thus there are 5 pairs in the month; in this month 3 pairs are pregnant and in the fourth month there are 8 pairs, of which 5 pairs bear another 5 pairs; these are added to the 8 pairs making 13 pairs in the fifth month; these 5 pairs that are born in this month do not mate in this month, but another 8 pairs are pregnant, and thus there are in the sixth month 21 pairs; to these are added the 13 pairs that are born in the seventh month; there will be 34 pairs in this month; to this are added the 21 pairs that are born in the eighth month; there will be 55 pairs in this month; to these are added the 34 pairs that are born in the ninth month; there**

(Continued)

(Continued)

Ninth 89 Tenth 144 Eleventh 233 Twelfth 377	**will be 89 pairs in this month; to these are added again the 55 pairs that are both in the tenth month; there will be 144 pairs in this month; to these are added again the 89 pairs that are born in the eleventh month; there will be 233 pairs in this month. To these are still added the 144 pairs that are born in the last month; there will be 377 pairs and this many pairs are produced from the above-written pair in the mentioned place at the end of one year.** **You can indeed see in the margin how we operated, namely that we added the first number to the second, namely the 1 to the 2, and the second to the third and the third to the fourth and the fourth to the fifth, and thus one after another until we added the tenth to the eleventh, namely the 144 to the 233, and we had the above-written sum of rabbits, namely 377 and thus you can in order find it for an unending number of months.”**

The chart below summarizes this calculation.



Month	Pairs	Number of pairs of adults (A)	Number of pairs of babies (B)	Total pairs
January 1		1	0	1
February 1		1	1	2
March 1		2	1	3
April 1		3	2	5
May 1		5	3	8
June 1		8	5	13
July 1		13	8	21
August 1		21	13	34
September 1		34	21	55
October 1		55	34	89
November 1		89	55	144
December 1		144	89	233
January 1		233	144	377

The number of pairs of mature rabbits living each month determines the Fibonacci sequence (column 1): 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, . . .

Nature is full of examples of the Fibonacci sequence. Flowers of every sort have petals in the count of 5, 8, and even the popular "she loves me, she loves me not" daisy usually sports 34 petals (Figure 1.3). Knowing that even number, you should always begin with the situation you do ***not*** want to end with. Only an odd number of petals, such as 33 or 35, will produce the situation you start with. The spiral of a seashell is the shape of a spiral formed by the Fibonacci numbers (Figure 1.4).

Figure 1.3

Source: Wikimedia Commons

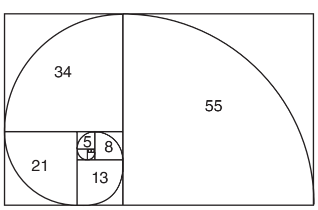
A closely aligned integer sequence was introduced by Eduardo Lucas (1842–1891). His sequence began with the numbers 1, 3 instead of 1, 1, which is the beginning of the Fibonacci numbers. Therefore, given $L_1 = 1$, $L_2 = 3$, using the "familiar" rule $L_{n+2} = L_n + L_{n+1}$, the sequence of Lucas numbers would be 1, 3, 4, 7, 11, 18, 29, 47, . . .

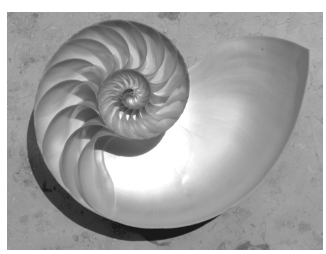
The Lucas numbers are related to the Fibonacci numbers by several identities, one of which appears below:

$$L_n = F_{n-1} + F_{n+1} = F_n + 2F_{n-1}$$

Mathematicians worldwide are still fascinated by the Fibonacci number sequence and continue to do research work on the sequence and related mathematics, producing university-level articles that are reviewed and published by *the Fibonacci Quarterly*, the official publication of the Fibonacci Association. *The Fibonacci Quarterly* is a scientific journal that has been publishing articles since 1963. Research articles as well as those presenting and solving elementary and advanced problems in related fields dealing with

Figure 1.4





Source: Chris 73/Wikimedia

Lucas numbers, primes, the golden ratio, graph coloring, the Pythagorean triples, and topics in advanced mathematics, music, computers, and art are submitted to the journal editorial staff for distribution to interested readers.

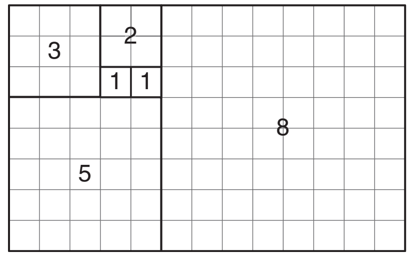
8. WHAT IS THE GOLDEN RATIO?[2]

Students may be familiar with the golden rectangle and the architectural beauty of the ratio of its sides. Extending these wonders, have the students take the ratio of any pair of consecutive Fibonacci numbers and consider their value. As the students use the larger Fibonacci numbers to calculate the ratio, they will find themselves getting closer and closer to the "ideal" ratio found in the length/width ratio of the golden rectangle, which is known as the *golden ratio*.

Ratio of Consecutive Fibonacci Numbers	Value of Ratio
$\frac{5}{3}$	1.66667
$\frac{8}{5}$	1.60000
$\frac{13}{8}$	1.62500
$\frac{21}{13}$	1.61538
$\frac{34}{21}$	1.619048
$\frac{233}{144}$	1.618056
$\frac{987}{610}$	1.618033
$\frac{f_{n+1}}{f_n}$	1.6180339887498948482045868 . . .

Furthermore, the Fibonacci numbers can be used as square blocks to build a close approximation of a golden rectangle, which is shown in Figure 1.5.

Figure 1.5



When we talk about the beauty of mathematics, we tend to think of the most beautiful rectangle. This is the golden rectangle, which has been shown by psychologists to be the most esthetically pleasing rectangle. We will now consider this golden ratio from the algebraic point of view.

Begin by having students recall the golden ratio: $\frac{1-x}{x} = \frac{x}{1}$.

This gives us $x^2 + x - 1 = 0$, and $x = \frac{\sqrt{5}-1}{2}$.

We let $\frac{\sqrt{5}-1}{2} = \frac{1}{\phi}$.

Not only does $\phi \cdot \frac{1}{\phi} = 1$ (obviously!), but $\phi - \frac{1}{\phi} = 1$.

This is the only number for which this is true.

Your students may want to verify this.[3]

By the way, students may want to know what value ϕ has. They can easily determine it with the help of a calculator:

$$\phi = 1.6160339887498948482045868343656381177203091798057 6\ldots$$

and $\frac{1}{\phi}$ = .61603398874989484820458683436563811772030917980576 . . .

There are many other interesting features of ϕ. Your students ought to be guided to develop some after you give them the proper hints. They might want to show that this infinite continued fraction has the value ϕ.

$$\phi = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}}}}}}$$

To do this, students ought to realize that nothing is lost by truncating the continued fraction at the first numerator. This will give them the following:

$\phi = 1 + \frac{1}{\phi}$, which yields the golden ratio.

Another curious relationship is

$$\phi = \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\cdots}}}}}}}}$$

Each of these is easily verifiable and can be done with a similar technique. We shall do the second one here and leave the first one to be justified by your students.

$$x = \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\cdots}}}}}}}}$$

$$x^2 = 1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\cdots}}}}}}}}$$

$$x^2 = 1 + x$$

$x = \phi$ from the definition of ϕ.

It is fascinating to observe what happens when we find the powers of ϕ.

$$\phi^2 = \left(\frac{\sqrt{5}+1}{2}\right)^2 = \frac{\sqrt{5}+3}{2} = \frac{\sqrt{5}+1}{2} + 1 = \phi + 1$$

$$\phi^3 = \phi \cdot \phi^2 = \phi(\phi+1) = \phi^2 + \phi = (\phi+1) + \phi = 2\phi + 1$$

$$\phi^4 = \phi^2 \cdot \phi^2 = (\phi+1)(\phi+1) = \phi^2 + 2\phi + 1 = (\phi+1) + 2\phi + 1 = 3\phi + 2$$

$$\phi^5 = \phi^3 \cdot \phi^2 = (2\phi+1)(\phi+1) = 2\phi^2 + 3\phi + 1 = 2(\phi+1) + 3\phi + 1 = 5\phi + 3$$

$$\phi^6 = \phi^3 \cdot \phi^3 = (2\phi+1)(2\phi+1) = 4\phi^2 + 4\phi + 1 = 4(\phi+1) + 4\phi + 1 = 8\phi + 5$$

$$\phi^7 = \phi^4 \cdot \phi^3 = (3\phi+2)(2\phi+1) = 6\phi^2 + 7\phi + 2 = 6(\phi+1) + 7\phi + 2 = 13\phi + 8$$

and so on.

A summary chart reveals a pattern among the coefficients of ϕ.

$$\phi^2 = \phi + 1$$
$$\phi^3 = 2\phi + 1$$
$$\phi^4 = 3\phi + 2$$
$$\phi^5 = 5\phi + 3$$
$$\phi^6 = 8\phi + 5$$
$$\phi^7 = 13\phi + 8$$

These are the Fibonacci numbers (see page 14).

By this time, your students are probably thinking that there is no end to the connections that one can draw to the golden ratio. Indeed, they are correct!

9. IS THERE A SMALLEST NUMBER, AND IS THERE A LARGEST NUMBER?

Before we can find the smallest or largest natural number, we have to define what we mean by natural number. Here we have two opinions. Some mathematicians define $\mathbb{N}$, the set of natural numbers, as the set of nonnegative integers, while others call only the positive integers the natural numbers. If we take the first definition, then $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ and the smallest natural number is zero. If we take the second definition, then $\mathbb{N} = \{1, 2, 3, 4, \ldots\}$ and the smallest natural number is one. In either case, there is a definite number we can identify as the smallest in the set of natural numbers.

If we now look for the largest number in the set $\mathbb{N}$, using either definition, we see that no matter what natural number we pick, there will always be a larger one as the members of the natural numbers differ from each other by one; thus, it is easy to create "the next natural number." So if you say that "n" is the largest, one can counter that by creating the natural number "$n + 1$" and $n + 1 > n$, and we now have found a number that is definitely bigger than n. Therefore, there is no largest natural number. When this happens, we say that the set goes on forever or that its members increase without bound. When considering a largest value in sets that have this concept of *unboundedness*, we note a largest value by the symbol $+\infty$, which is translated as *positive infinity* and means that the set has no identifiable value as the largest, as the members of the set continue to grow in a positive direction. Think of standing on the natural number line and looking toward the right where the "horizon" lies. Looking toward the horizon, you begin to walk in that direction while focused on reaching the horizon. But you never actually get there since the horizon continues to stretch to the right, beyond where you are.

It would be the same as we turn our attention to the set of positive real numbers, $\mathbb{R}^{positive}$, and look for a largest member. We see that the set $\mathbb{R}^{positive}$, as with the set $\mathbb{N}$, grows without an upper bound, and for every real number, there will always be a larger one. We say the upper limit of $\mathbb{R}^{positive}$ is also $+\infty$, which implies that the positive real numbers increase without bound.

Continuing, we look in the other direction to see if the set of positive real numbers, $\mathbb{R}^{positive}$, has a smallest member, and we meet the concept of *boundedness*. Before we begin our search, consider this task: You are standing about 20 feet from the door. The door is point "0" and you must move toward that door by halving your distance from it in each move. Your first move brings you to 10 feet from the door, then 5 feet, then 2.5 feet, then 1.25 feet, and now you are close. Your next move brings you to .625 feet (almost $7\frac{1}{2}$ inches) from the door, and then you are .3125 feet (less than 4 inches) from the door. Watch out that you do not bump your nose as you move this close to the "zero" point! In theory, you get closer and closer to that door but you never really will be at the exact zero place—the door.

We can now form an analogy with the positive real numbers. The set, $\mathbb{R}^{positive}$, has a lower bound of zero that is *not* a member of this set since zero is in a class by itself and is neither positive nor negative. The positive numbers begin to the right of the zero point. As we find smaller and smaller positive real numbers, we are moving closer toward zero. We can always get a bit closer by halving the smallest real number we have, but we will never hit the zero. So there is no smallest positive real number.

Moving on, let us consider the smallest number in the set of all the real numbers, $\mathbb{R}$, which would include all the negative numbers and zero, along with all the positive numbers. As the positive real numbers increase without bound, the negative real numbers decrease without bound. Taking all the real numbers into consideration, for every real number, you can find one less than it by just subtracting any positive amount from it. So if one claims that x is the smallest real number, you can offer $x - .001$ as smaller: $x - .001 < x$. Likewise, looking for the largest real number, if one claims that y is the largest, then you can offer $y + .001$ as a larger number. So the set of real numbers, $\mathbb{R}$, increases and decreases without bounds and ranges from "that horizon off to the left" known as negative infinity, $-\infty$, to "that horizon off to the right" known as positive infinity, $+\infty$. In interval notation, we write that $\mathbb{R} = (-\infty, +\infty)$. This is an "open" interval, meaning the $+\infty$ and the $-\infty$ represent the concept of the set of real values, having no bounds in either direction and having no smallest and no greatest real number.